

The Tate Thomason Conjecture

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Abstract

We prove the Tate Thomason conjecture using $\mathcal{K}_{(l)} \wedge M(Z/l)$ localized spectra where \mathcal{K} is the complex topology spectrum, and $M(Z/l)$ the Moore spectrum at Z/l . Fundamental to our proof is Theorem 2.1 below, where we show that certain $\mathcal{K}_{(l)} \wedge M(Z/l)$ localized spectra are $\mathcal{K}_{(l)}$ module spectra. We also make use of the notion of etale K Theory.

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Introduction:

By a spectrum we mean the following: A spectrum X is a collection of simplicial sets X_n for $n \geq 0$ together with morphisms of simplicial sets $\sigma_n : \Sigma X_n \rightarrow X_{n+1}$. A morphism of spectra $f : X \rightarrow Y$ is a collection of morphisms $f_n : X_n \rightarrow Y_n$ of simplicial sets that commute with the structure maps σ_n , ie $\sigma_n \circ \Sigma f_n = f_{n+1} \circ \sigma_n$. We are going to consider the stable homotopy category \mathcal{S} derived from the Bousfield-Friedlander model category of simplicial spectra [BF78]. The l -localized periodic topological K spectrum with period two is written as $\mathcal{K}_{(l)}$. It decomposes as $\mathcal{K}_{(l)} = \bigvee_{i=1}^{l-2} \Sigma^{2i} E(1)$ involving certain spectrum $E(1)$ defined below and in [B83], belonging to the stable homotopy category \mathcal{S} . Here l is an odd prime. The topological K spectrum is written \mathcal{K} and its localization is written L . Since $\mathcal{K}_{(l)}$ equivalences in \mathcal{S} are the same as $E(1)$ equivalences, it follows that in \mathcal{S} , $\mathcal{K}_{(l)}$ localizations are the same as $E(1)$ localizations. Given W in \mathcal{S} , we denote this equivalent localizations by $L_{E(1)}W$. The full subcategory of $E(1)$ -local spectra in \mathcal{S} is equivalent to the homotopy category of certain localization of the Bousfield Friedlander model category [CR07]. The spectra we consider are $K(X)$ where X is a scheme over a finite field F_q with $q = p^s$ where $p \neq l$, and $K(X)$ is Quillen's Algebraic K theory spectra related to X . Let X be a spectrum. Then the $E(1)_*E(1)$ comodule $E(1)_*(X)$ is an object in $\mathcal{B}(l)_*$, a category defined below and also in [B83] and

[CR07]. If Y is a $E(1)$ -module spectrum then $\pi_*(Y)$ is a $\pi_*(E(1))$ module and $\mathcal{U}(\pi_*(Y)) = E(1)_*(Y)$ in $\mathcal{B}(l)_*$, where \mathcal{U} is the universal functor defined in [B83]. We know that $\pi_*(E(1)) \simeq Z_{(l)}[\nu, \nu^{-1}]$. Also $E(1)_*E(1)$ is zero if $* \neq 0 \bmod(l)$ and $E(1)_0E(1) = \Lambda$ where Λ is the ring $Z_l[[t]]$ and $E(1)_0E(1)$ is the l -adic completion of the $Z_{(l)}$ -module $E(1)_0E(1)$. Our aim is to prove the Tate Thomason conjecture ([TH89]): Let X_∞ be $X \times_{\text{Spec}(F_q)} \text{Spec}(\bar{F}_q)$, where X is a smooth projective variety over $\text{Spec}(F_q)$, and \bar{F}_q is the algebraic closure of F_q , then the homotopy group $\pi_{-1}(L_{E(1)}K(X_\infty))$ is reduced. See [TH89] page 390 diagram 21. Through Thomason's descent theorem [TH85] see also [M96], this conjecture implies the Tate conjecture for odd primes l . The proof is based on a theorem by Bousfield [B83] which is theorem 1.1 of section 1 and also on theorem 2.1, corollary 2.1 and theorem 2.3 from section 2. Corollary 2.1 states basically that $L_{E(1)/l}K(X_\infty)$ is a $\widehat{E(1)}$ module spectrum, with $E(1)/l = E(1) \wedge M(Z/l)$, where $M(Z/l)$ is the Moore spectrum at Z/l .

0. The Tate Conjecture

For a projective smooth variety X over a field k , the i -th Chow Group $CH^i(X)$ is generated by cycles of codimension i on X modulo rational equivalence. If l is a prime invertible in k we can define a Q_l -linear cycle morphism from the i -th Chow group into the $2i$ -th l -adic étale cohomology with i -th Tate twist coefficient:

$$\gamma^i(X)_{Q_l} : CH^i(X) \otimes Q_l \mapsto H^{2i}(X, Q_l(i))$$

Let k_1 be a finite subfield of \bar{F}_q . Let $H^{2i}(X, Q_l(i))^{Gal(\bar{F}_q/k_1)}$ be the subspace of $H^{2i}(X, Q_l(i))$ fixed by $Gal(\bar{F}_q/k_1)$, then the Tate Conjecture states:

0.1 Tate Conjecture: *The image of the cycle class morphism $\gamma^i(X)_{Q_l}$ in the cohomology group $H^{2i}(X, Q_l(i))$ is exactly the union of the subspaces $H^{2i}(X, Q_l(i))^{Gal(\bar{F}_q/k_1)}$, where k_1 ranges over all finite subfields of \bar{F}_q*

0.2 Thomason's reformulation of the Tate Conjecture:

Take the smooth projective variety X_∞ . Let $X_n = X \otimes F_{q^n}$ where X is a smooth projective variety over F_q as mentioned in the introduction. Just for this section let \mathcal{K} be the complex topology spectrum localized at the prime l which is usually noted \mathcal{K}_l . Thomason proves in [Th89] the following theorem:

Theorem 0.1 (Tate Thomason's Conjecture): *The Tate Conjecture for X_∞ is equivalent to the finiteness statement that for all $n \in \mathbb{N}$, See [TH89]*

$$\mathrm{Hom}(Q/Z_{(l)}, \pi_{-1}L_{\mathcal{K}}K(X_n)) = 0$$

Definition 0.1: A group G which verifies that $\mathrm{Hom}(Q/Z_{(l)}, G) = 0$ is said to be l reduced.

We will simplify all over this work the terminology by saying that a group is reduced when we really mean that it is l -reduced.

The main lemmas stated by Thomason in [Th89] to prove theorem 0.1 are the following:

Lemma 0.1: *Let X be a smooth projective variety over F_q with $q = p^n$ and p prime. If l is a prime number different from p , and if $K^{Top}(X)$ is in Thomason's notation the topological K -theory spectrum, and $(\dots)^\wedge$ is the l -adic completion of a spectrum, then*

$$L_{\mathcal{K}}K(X)^\wedge \cong K^{Top}(X)^\wedge$$

The proof of this lemma follows from Thomason's descent theorem proved in [TH85, 4.1] which relates algebraic K -theory to topological K -theory. See also ([TH89, page 388, equation (14) and references therein)

Remark 0.0: It also follows from Thomason's descent theorem that

$$K^{Top}/l^\nu(X) \cong L_{\mathcal{K}}(X)/l^\nu$$

See the descent problem in ([M96], section 4)

Lemma 0.2: (See [TH89], the lemma of page 387)

The image of $\text{colim} K_0^{Top}(X_n) \hat{\otimes} Q$ in $K_0^{Top}(X_\infty) \hat{\otimes} Q$ consists precisely of those elements of finite orbit under $\text{Gal}(\bar{F}_q/F_q)$.

Remark 0.1: (See [TH89, page 387 square diagram 10]) Consider the square

$$\begin{array}{ccc} \text{colim}(K_0(X_n) \otimes Q_l) & \longrightarrow & \text{colim}(K_0^{Top}(X_n) \hat{\otimes} Q) \\ \downarrow & & \downarrow \\ K_0(X_\infty) \otimes Q_l & \longrightarrow & K_0^{Top}(X_\infty) \hat{\otimes} Q \end{array}$$

The left vertical arrow is an isomorphism and in light of this, lemma 0.2, and the fact that $K_0(X_\infty) \otimes Q_l$ is isomorphic to $\bigoplus_{i=1}^d CH^i(X_\infty) \otimes Q_l$ with d the dimension of X , and $K_0^{Top}(X_\infty) \hat{\otimes} Q$ is isomorphic to $\bigoplus_{i=1}^d H_{et}^{2i}(X_\infty, Q_l(i))$, (see [TH89, page 390]) the bottom map is exactly the cycle map $\gamma(X_\infty)_{Q_l}$ and the Tate Conjecture is equivalent to the conjecture that $K_0(X_\infty) \otimes Q_l$ and $\text{colim}(K_0^{Top}(X_n) \hat{\otimes} Q)$ have the same image in $K_0^{Top}(X_\infty) \hat{\otimes} Q$.

Lemma 0.3: *There is an exact sequence*

$$\text{colim}(K_0(X_n) \otimes Q_l) \mapsto \text{colim}(K_0^{Top}(X_n) \hat{\otimes} Q) \mapsto \text{colim} \text{Hom}(Q/Z_{(l)}, \pi_{-1}(L_K K(X_n)) \otimes Q) \mapsto 0$$

The proof uses lemma 0.1, and the arguments of [TH89], pp 389-390. It follows by virtue of lemma 0.2 and remark 0.1 that if $\text{Hom}(Q/Z_{(l)}, \pi_{-1}(L_K K(X_n))) = 0$ for all $n \in N$ then the Tate Conjecture is true, and this is exactly the statement of theorem 0.1.

Corollary 0.1 *If $\text{Hom}(Q/Z_{(l)}, \pi_{-1} L_K K(X_\infty)) = 0$, then the Tate Conjecture is true.*

The proof of this corollary follows from the diagram (21) on ([TH89], page 390), which is an extension of the diagram of Remark 0.1, and the arguments therein.

Remark 0.2: To our knowledge, Thomason never said anything about the l -reducibility of the homotopy group $\pi_{-1} L_K K(X_\infty)$. At this point is where our work starts, following section 1, where the most important concepts of

Bousfield's work [B83] are outlined. Our aim is to prove the hypothesis of corollary 0.1.

1. The Spectrum $E(1)$ and the category $\mathcal{B}(l)_*$

1.1 The Category $\mathcal{B}(l)_*$

We begin by describing an abelian category, denoted $\mathcal{B}(l)_*$, equivalent to the category of $E(1)_*E(1)$ -comodules (see [B83], 10.3) Bousfield describes $\mathcal{B}(l)_*$ as follows: Let l be an odd prime and let \mathcal{B} denote the category of $Z_{(l)}[Z_{(l)}^*]$ -modules for the group ring $Z_l[Z_l^*]$, where $Z_{(l)}^*$ are the units in $Z_{(l)}$, with the action by the group ring defined by Adams operations $\Psi^k : M \mapsto M$ which are automorphisms and satisfy the following:

- i) There is an eigenspace decomposition

$$M \otimes Q \cong \bigoplus_{j \in Z} W_{j(l-1)}$$

such that for all $w \in W_{j(l-1)}$ and $k \in Z_{(l)}$,

$$(\Psi^k \otimes id)w = k^{j(l-1)}w$$

- ii) For all $x \in M$ there is a finitely generated submodule $C(x)$ containing x , satisfying: for all $m \geq 1$ there is an n such that the action of $Z_{(l)}^*$ on $C(x)/l^m C(x)$ factors through the quotient of $(Z/l^{n+1})^*$ by a subgroup of order $l-1$.

To build the category $\mathcal{B}(l)_*$ out of the above category \mathcal{B} , we additionally need the following:

Let $T^{j(l-1)} : \mathcal{B} \mapsto \mathcal{B}$ with $j \in Z$ denote the following equivalence:

For all M in \mathcal{B} , $T^{j(l-1)}(M) = M$ as $Z_{(l)}$ -module, but not as $Z_{(l)}[Z_{(l)}^*]$ -module since the Adams operations in $T^{j(l-1)}(M)$ are now $k^{j(l-1)}\Psi^k : M \mapsto M$ where Ψ^k is the Adams operation of multiplication by k in \mathcal{B} . Now an object in $\mathcal{B}(l)_*$

is defined as a collection of modules $M = (M_n)_{n \in \mathbb{Z}}$, with M_n in \mathcal{B} together with a collection of isomorphisms for all $n \in \mathbb{Z}$,

$$T^{l-1}(M_n) \mapsto M_{n+2(l-1)}$$

Note that the category \mathcal{B} can be viewed as the subcategory of $\mathcal{B}(l)_*$ consisting of those objects $(M_n)_{n \in \mathbb{Z}}$ such that $M_n = M$ if n is congruent to 0 mod $2(l-1)$ and 0 otherwise

In [B83] Bousfield constructs a functor $\mathcal{U}: \pi_*(E(1) - \text{Mod}) \mapsto \mathcal{B}_*$. For $H \in \pi_*(E(1) - \text{Mod})$, let \mathcal{U} in \mathcal{B} consist of the objects $\mathcal{U}(H_n)$ in \mathcal{B} for all $n \in \mathbb{Z}$.

We know from ([B83]page 929) that $\mathcal{U}: \pi_*(E(1) - \text{Mod}) \mapsto \mathcal{B}(l)_*$ verifies:

$$\mathcal{U}(G) = E(1)_*E(1) \otimes_{\pi_*E(1)} G$$

for all $\pi_*(E(1))$ -module G . In particular taking 0 component, $\mathcal{U}_0(G) = E_0(1)E(1) \otimes_{Z(l)} G$

In [B83] the following theorem (which will be crucial for us) is proved:

Theorem 1.1 (Bousfield): *For each $E(1)$ -module spectrum Y in \mathcal{S} , there exists a map $m: E(1)_*(Y) \mapsto \mathcal{U}(\pi_*(Y))$ which is an isomorphism in $\mathcal{B}(l)_*$.*

1.2 The Spectrum $E(1)$ and its homology theory $E(1)_*$.

Given $E(1)$, which by construction depends on the prime l , there is a map $E(1) \mapsto \mathcal{K}_l$ which is a ring morphism (see [R] Chapter VI Theorem 3.28) and verifies the equivalence $\mathcal{K}_{(l)} = \bigvee_{i=1}^{l-2} \Sigma^{2i} E(1)$. There are Adams operations $\Psi^k: E(1) \mapsto E(1)$ with k in Z_l^* which are the units in Z_l . These Adams operations are ring spectra equivalences and Ψ^k carries ν^j to $k^{j(l-1)}\nu^j$ in $\pi_{2j(l-1)}E(1)$ for each integer j where ν is such that $\pi_*E(1) = Z_{(l)}[\nu, \nu^{-1}]$ and ν has degree $2(l-1)$. Another property of $E(1)$ is that $E(1)$ localization is the same as $calK_{(l)}$ localization.

The homology $E(1)_*(X)$ with X a spectrum also has Adams operations $\Psi^k: E(1)_*(X) \mapsto E(1)_*(X)$. One checks that $\Psi^k(\nu^j x) = k^{j(l-1)}\nu^j \Psi^k(x)$ for each integer j and k in $Z_{(l)}^*$ and $x \in E(1)_*(X)$. The multiplication by ν^j induces an isomorphism $\nu^j: T^{j(l-1)}E(1)_n(X) \mapsto E(1)_{n+2j(l-1)}(X)$ in $\mathcal{B}(l)_*$ for

each $j, n \in \mathbb{Z}$. It follows that $E(1)_*(X)$ is in $\mathcal{B}(l)_*$ for each spectrum X in \mathcal{S} by taking $E(1)_n(X) = M_n$ defined in 1.1 and by taking as Adams operations, the Adams operations just mentioned.

Remark 1.0: In [B83] it is shown that the Adams operations in \mathcal{B} are all canonically determined by a single operation Ψ^r where r is a fixed integer and is a generator of the group which is the quotient of $\mathbb{Z}/l^2\mathbb{Z}$ by its subgroup of order $l-1$. Let $\Psi = \Psi^r - 1$. Given M in \mathcal{B} define M^Ψ and M_Ψ as the kernel and cokernel of $\Psi : M \mapsto M$.

Remark 1.1: Given l^ν , with $\nu \in \mathbb{N}$, and $M(Z/l^\nu)$ the Moore spectrum of the ring \mathbb{Z}/l^ν , and if X is a smooth variety over a field k where $k = F_{q^n}$ or $k = \bar{F}_q$ with $q = p^n$ and L is the localization functor at the complex K-Theory spectrum \mathcal{K} , then

$$LK(X)/l^\nu = LK(X) \wedge M(Z/l^\nu) = K(X_n) \wedge L\Sigma^\infty S^o \wedge M(Z/l^\nu) = K(X)/l^\nu \wedge L\Sigma^\infty S^o = L(K(X)/l^\nu). \text{ The same argument applies to } L_{E(1)}.$$

Remark 1.2: $\pi_*(L(K(X)/l^\nu))$ is l -torsion and this statement is also true with L interchanged with $L_{E(1)}$.

The claim follows from the exact sequence

$$0 \mapsto \pi_*(LK(X) \otimes \mathbb{Z}/l^\nu) \mapsto \pi_*(LK(X) \wedge M(Z/l^\nu)) \mapsto \text{Tor}^1(\pi_{*-1}(LK(X)), \mathbb{Z}/l^\nu) \mapsto 0$$

which splits (See [TH85] Appendix A) and by remark 1.3.

2. Main Theorems.

Let $\widehat{E(1)} = L_{M(\mathbb{Z}/l\mathbb{Z})}(E(1))$ be the l -adic completion of the spectrum $E(1)$. We now come to the theorem:

Theorem 2.1: *Let $K(1) = E(1)/l = E(1) \wedge M(\mathbb{Z}/l\mathbb{Z})$ with l an odd prime not equal to p where $q = p^n$ and $X_\infty = X \otimes \bar{F}_q$ with X a smooth variety over F_q , then $L_{K(1)}K(X_\infty)$ is an $\widehat{E(1)}$ -module spectrum.*

Proof: By a result in ([M75] Th 2.8 Chap VIII, page 218) there is a unique isomorphism of ring spectra $\widehat{K\bar{F}_q} \mapsto \widehat{ku}$, where $\widehat{K\bar{F}_q}$ and \widehat{ku} are respectively the l -adic completions of algebraic K-theory spectrum of \bar{F}_q and of the connective cover of the spectrum of topological K-theory.

Localizing with L , the localization functor of complex K-theory we get an isomorphism of ring spectra $\widehat{\mathcal{K}} \mapsto L_{\mathcal{K}}L_{M(\mathbb{Z}/l\mathbb{Z})}K(\bar{F}_q)$ since $L_{M(\mathbb{Z}/l\mathbb{Z})}K(\bar{F}_q) = \widehat{K(\bar{F}_q)}$ and $\widehat{\mathcal{K}} = L_{M(\mathbb{Z}/l\mathbb{Z})}L_{\mathcal{K}}ku$.

Localizing further at l we obtain a ring map isomorphism

$$(1)\widehat{\mathcal{K}} \mapsto L_{K(1)}K(\bar{F}_q)$$

since $L_{K(1)} = L_{M(Z/lZ) \wedge E(1)} = L_{M(Z/lZ)}L_{E(1)}$ and $L_{E(1)} = L_l L_{\mathcal{K}}$ where L_l is the localization functor at l .

Now $K(X_\infty)$ is a $K\bar{F}_q$ -module spectrum (See [DM95] page 13) and so $L_{K(1)}K(X_\infty)$ is a $L_{K(1)}K\bar{F}_q$ -module spectrum ([DM95] page 14). Then by (1) we have that $L_{K(1)}K(X_\infty)$ is a $\widehat{\mathcal{K}}$ module spectrum. Since the ring map $E(1) \mapsto \mathcal{K}_l$ implies that $\widehat{E(1)} \mapsto \widehat{\mathcal{K}}$ is a ring map of spectra, then we conclude that $L_{K(1)}K(X_\infty)$ is a $\widehat{E(1)}$ module spectrum.

Remark 2.1: The localization functor $L_{K(1)}$ is equivalent to the localization functor of $\widehat{E(1)}^*$ cohomology (see [DM95], 4.10 page 15) so that theorem 2.1 is also proven in [DM95, 5.1 page 20] and moreover all results stated on that reference about this cohomology functor are also valid for the functor $L_{K(1)}$.

Corollary 2.1: $L_{K(1)}(K(X_\infty))$ is an $E(1)$ -module spectrum if the prime l which defines $K(1)$ is not equal to p .

Proof: This is so since the map $E(1) \mapsto \widehat{E(1)}$ is a localization map and is therefore a ring map of spectra, (see [CG05])

Corollary 2.2 $L_{K(1)}(K(X_\infty)/l^\nu)$ is an $E(1)$ -module spectrum

Proof: We use a localization argument as in Corollary 2.1 and the remark 1.3

We conjecture

Conjecture 2.0: $L_{E(1)}(K(X_\infty))$ is an $E(1)$ -module spectrum if the prime l which defines $E(1)$ is not equal to p

Corollary 2.3: the functor \mathcal{U} introduced in section 1 verifies,

$$\mathcal{U}(\pi_*(L_{K(1)}(K(X_\infty)/l^\nu))) \cong E(1)_*(L_{K(1)}(K(X_\infty)/l^\nu)).$$

Proof: Follows immediately from Theorem 1.1, and Corollary 2.2

Theorem 2.2: $\pi_{-1}(L_{K(1)}(K(X_\infty)/l^\nu))$ is reduced.

Proof: By corollary 2.3 and the arguments of ([B83], Prop 6.6 pages 913 and 914) since $\pi_*(L_{K(1)}(K(X_\infty)/l^\nu))$ is a Z_l -module with l -torsion we have that

$$\pi_*(L_{K(1)}(K(X_\infty)/l^\nu)) \cong (\mathcal{U}(\pi_*(L_{K(1)}(K(X_\infty)/l^\nu))))^\Psi \cong (E(1)_*(L_{K(1)}(K(X_\infty)/l^\nu)))^\Psi.$$

Hence the theorem follows if we prove that $(E(1)_{-1}(L_{K(1)}(K(X_\infty)/l^\nu)))^\Psi$ is reduced. It suffices to prove that $(E(1)_{-1}(L_{K(1)}(K(X_\infty)/l^\nu)))$ is reduced. We must show that $Q/Z_{(l)}$ is not included in $M := (E(1)_{-1}(L_{K(1)}(K(X_\infty)/l^\nu)))$, since if M contains a divisible subgroup, this divisible subgroup, being l -torsion, must be a direct sum of $Q/Z_{(l)}$.

Suppose that $Q/Z_{(l)}$ is included in $(E(1)_{-1}(L_{K(1)}(K(X_\infty)/l^\nu)))$. Then, it would be included in all $E(1)_m(L_{K(1)}(K(X_\infty)/l^\nu))$ for any integer m . This is so since as we explained in section 1, $T^{j(m-1)}(E(1)_{-1}(L_{K(1)}(K(X_\infty)/l^\nu)))$ which is the same object as $(E(1)_{-1}(L_{K(1)}(K(X_\infty)/l^\nu)))$ but with different Ψ^k operations, is isomorphic to $E(1)_{-1+2j(m-1)}(L_{K(1)}(K(X_\infty)/l^\nu))$. The isomorphism is a consequence of the object $E(1)_*(L_{K(1)}(K(X_\infty)/l^\nu))$ being in $\mathcal{B}(l)_*$ as explained in section 1. Then, by corollary 2.3, and since $\mathcal{U}(G)$ is an infinite direct sum of copies of G if G is an l -torsion abelian group (see [B83], page 911) $Q/Z_{(l)}$ would be included in an infinite direct sum of copies of $\pi_m(L_{K(1)}(K(X_\infty)/l^\nu))$ for any integer m , so that,

$$\text{Hom}(Q/Z_{(l)}, Q/Z_{(l)}) \neq 0 \subseteq \text{Hom}(Q/Z_{(l)}, \bigoplus_{i=1}^{\infty} \pi_m(L_{K(1)}(K(X_\infty)/l^\nu)))$$

$$\text{Hom}(Q/Z_{(l)}, \bigoplus_{i=1}^{\infty} \pi_m(L_{K(1)}(K(X_\infty)/l^\nu))) \subseteq \text{Hom}(Q/Z_{(l)}, \Pi_{i=1}^{\infty} \pi_m(L_{K(1)}(K(X_\infty)/l^\nu)))$$

and

$$\text{Hom}(Q/Z_{(l)}, \Pi_{i=1}^{\infty} \pi_m(L_{K(1)}(K(X_\infty)/l^\nu))) = \Pi_{i=1}^{\infty} \text{Hom}(Q/Z_{(l)}, \pi_m(L_{K(1)}(K(X_\infty)/l^\nu)))$$

Hence, to finish the proof of this lemma, it is sufficient to show that for some integer m , $\pi_m(L_{K(1)}(K(X_\infty)/l^\nu))$ is reduced since if that is the case $\text{Hom}(Q/Z_{(l)}, \pi_m(L_{K(1)}(K(X_\infty)/l^\nu))) = 0$ and we arrive to a contradiction. But this is so, by the following remark:

Remark 2.1: Given an integer $m \geq 1$, $\pi_m(L_{K(1)}(K(X_\infty)/l^\nu))$ is reduced if $\pi_m(L_{E(1)}(K(X_\infty)/l^\nu))$ is of finite order, since $\pi_m(L_{K(1)}(K(X_\infty)/l^\nu)) = \pi_m([\widehat{L_{E(1)}(K(X_\infty)/l^\nu)}])$ where $[\widehat{L_{E(1)}(K(X_\infty)/l^\nu)}]$ is the l -adic completion of $L_{E(1)}(K(X_\infty)/l^\nu)$ and therefore have reduced homotopy groups which are equal to $\pi_m L_{E(1)}(K(X_\infty)/l^\nu) \otimes Z_l$ if $\pi_m(L_{E(1)}(K(X_\infty)/l^\nu))$ is of finite order.

Now X_∞ is a smooth projective variety over the field \bar{F}_q , and then the homotopy groups $\pi_m(L_{E(1)}(K(X_\infty)/l^\nu))$ are finite as stated in [RTH89 page

427]. This fact follows from the weak equivalence between $K^{Top}/l^\nu(X_\infty)$ and $H_{et}^*(X_\infty, K^{Top}/l^\nu(.))$ which is stated in ([TH85], page 520 after diagram 4.15). It can also be shown in the following way:

We have, by Thomason's descent theorem (Theorem 4.1, [TH85] see also appendix A page 547) that,

$$H_{et}^s(X_\infty, Z/l^\nu(j/2)) \longrightarrow \pi_{j-s}(L_{E(1)}(K(X_\infty)/l^\nu) = K_{j-s}(X_\infty)/l^\nu[\beta^{-1}]$$

(1)

where β is the Bott element. Now by [TH85, Th4.11 page 520], or [TH84, page 400], or [RTH89], $K_i(X_\infty)/l^\nu[\beta^{-1}]$ is isomorphic to etale K_i theory with $mod(l^\nu)$ coefficients. Now consider the spectral sequence which is the strongly convergent fourth-quadrant spectral sequence in [DwFr85 page 260]. Remember that for us X_∞ is a smooth projective variety, therefore X_∞ is proper ([Liu], page 108), and SGAIV Vol 3 page 145, (see also [M02] page 987) imply that the groups $H_{et}^s(X_\infty, Z/l^\nu(j/2))$ are finite. This spectral sequence collapses at E_2 modulo torsion of bounded order (see [TH84] and [Sou82]. The groups $K_m^{et}(X_\infty, Z/l^\nu)/F^p K_m^{et}(X_\infty, Z/l^\nu)$ are finite, (see [Sou82] page 295, Th 5, i)). The filtration $F^p K_m^{et}(X_\infty, Z/l^\nu)$ from the above spectral sequence is equal to the gamma filtration of [Sou82, 1.2 page 274 and page 288], ie $F^p K_m^{et}(X_\infty, Z/l^\nu) = F_\gamma^i K_m^{et}$ for $m + p = 2i$. The gamma filtration vanishes for i big enough, ie bigger than the dimension of X if we follow arguments similar to [Sou85, Th1, page 494]. This implies the finiteness of $\pi_m(L_{E(1)}(K(X_\infty)/l^\nu))$. To see the vanishing of the gamma filtration it is sufficient to consider $X_\infty = Spec(A)$ by [Sou82, page 280] since X is a smooth projective variety. Following [Sou85] we consider first a different gamma filtration defined by $G_\gamma^i K_m^{et}(A) = \{\gamma^i(x), x \in K_m^{et}(A)\}$. This gamma filtration vanishes for i big enough with proof similar to [Sou85], page 494]. In [Soule85], Soule considers in the study of the gamma filtration, applied to Quillen K theory, the following facts:

Stability of Volodin's K theory, $V(A)$, which is coincident with Quillen's K theory.

$$BGL_N(A) \mapsto BGL_{N+1}(A), V(A) = colim V_N(A), \Omega BGL(A)^+ \mapsto V(A)$$

while for the gamma filtration applied to etale K theory one must consider:

Stability of etale K Theory [Sou82, page 280],

$$BGL_N(A) \mapsto BGL_{N+1}(A), K^{et}(A) = colim BGL_N(F_q)^A, BGL(A)^+ \mapsto K^{et}(A)$$

see [Sou82, pages 280 and 281].

The vanishing of $G_\gamma^M(K_m^{et}(A))$ for M big enough, implies the vanishing of $F_\gamma^J(K_m^{et}(A)) = \{\gamma^{i_1}(x) \cdot \gamma^{i_2}(x) \dots \gamma^{i_n}(x), i_1 + i_2 + \dots i_n \geq j, x \in K_m^{et}(A)\}$ for J big enough. This can be seen in the following way:

Define $\gamma_t(x) = \sum_{i=1}^{\infty} \gamma^i(x) t^i$ for $x \in K_m^{et}(A)$. Then by the vanishing argument from above, we have $\gamma_t(x) = \sum_{i=1}^M \gamma^i(x) t^i$. Hence, since $\gamma_t(x) \gamma_t(-x) = \gamma_t(0) = 1$ (see [CW2], page 29), the coefficients of $\gamma_t(x)$ are nilpotent with a uniform bound for all $x \in K_m^{et}(A)$. In particular, $\gamma^1(x) = x$ is nilpotent for all $x \in K_m^{et}(A)$ with uniform bound. Hence by the above definition of $F_\gamma^J(K_m^{et}(A))$, for a product of the form $\gamma^{i_1}(x) \cdot \gamma^{i_2}(x) \dots \gamma^{i_n}(x)$, since $\gamma^i(x) = x \cdot F(x)$, for certain polynomial $F(x)$ [CW2, page 28], then either we get an expression x^N on that product, which vanishes, or we get an expression $\gamma^l(x)$ which vanishes, since N and l are big enough as a consequence that the upper index J from the gamma filtration is big enough.

Corollary 2.4: $\pi_m(L_{K(1)}(K(X_\infty)/l^\nu))$ is reduced for all integer m .

Proof: Following the same procedure as above we obtain theorem 2.2 for any integer m without any changes on the proofs.

Theorem 2.3: $\pi_{-1}(L_{K(1)}K(X_\infty))$ is reduced.

Proof: By theorem 2 we know that

$$Hom(Q/Z_{(l)}, \pi_{-1}(L_{K(1)}(K(X_\infty)/l^\nu))) = 0$$

By Corollary 2.4,

$$Hom(Q/Z_{(l)}, \pi_0(L_{K(1)}(K(X_\infty)/l^\nu))) = 0$$

and we have,

$$0 \mapsto \pi_0(L_{K(1)}K(X_\infty)) \otimes Z/l^\nu \mapsto \pi_0(L_{K(1)}K(X_\infty) \wedge M(Z/l^\nu)) \mapsto Tor^1(\pi_{-1}(L_{K(1)}K(X_\infty)), Z/l^\nu) \mapsto 0$$

with $Tor^1(\pi_{-1}(L_{K(1)}K(X_\infty)), Z/l^\nu)$ a direct summand of $\pi_0(L_{K(1)}K(X_\infty) \wedge M(Z/l^\nu))$.

Taking inverse limit with respect to the parameter ν with l fixed,

$$0 \mapsto \lim Tor^1(\pi_{-1}(L_{K(1)}K(X_\infty)), Z/l^\nu) \mapsto \lim \pi_0(L_{K(1)}K(X_\infty)/l^\nu)$$

where $M = \lim Tor^1(\pi_{-1}(L_{K(1)}K(X_\infty)), Z/l^\nu)$ is equal to

$$\Pi_{i=1}^\infty \{g_i = l^i - \text{torsion} - \text{element} \in \pi_{-1}(L_{E(1)}K(X_\infty)) : lg_{i+1} = g_i\}$$

By taking the left exact functor $Hom(Q/Z_{(l)}, -)$ in the above sequence we get,

$$0 \mapsto Hom(Q/Z_{(l)}, M) \mapsto Hom(Q/Z_{(l)}, \lim \pi_0(L_{K(1)}K(X_\infty)/l^\nu)) \mapsto \dots$$

Since,

$$Hom(Q/Z_{(l)}, \lim \pi_0(L_{K(1)}K(X_\infty)/l^\nu)) = \lim Hom(Q/Z_{(l)}, \pi_0(L_{K(1)}K(X_\infty)/l^\nu)) = 0$$

then

$$Hom(Q/Z_{(l)}, M) = 0$$

On the other hand consider,

$$0 \mapsto \pi_{-1}(L_{K(1)}K(X_\infty)) \otimes Z/l^\nu \mapsto \pi_{-1}(L_{K(1)}K(X_\infty)/l^\nu)$$

By taking the $Hom(Q/Z_{(l)}, -)$ we get, since $Hom(Q/Z_{(l)}, \lim \pi_{-1}(L_{K(1)}K(X_\infty)/l^\nu)) = \lim Hom(Q/Z_{(l)}, \pi_{-1}(L_{K(1)}K(X_\infty)/l^\nu)) = 0$

$$Hom(Q/Z_{(l)}, (\pi_{-1}(L_{K(1)}K(X_\infty)))^l) = 0$$

where $\pi_{-1}(L_{K(1)}K(X_\infty))^l$ is the l completion of $\pi_{-1}(L_{K(1)}K(X_\infty))$. Then, the image of an f in $Hom(Q/Z_{(l)}, \pi_{-1}(L_{K(1)}K(X_\infty)))$ must be included in G , where G is the intersection of all $l^\nu \pi_{-1}(L_{K(1)}K(X_\infty))$, since this intersection is exactly the kernel of the completion homomorphism.

Observe that by definition of M , each element of M has its components belonging to G .

Consider an f as above. Let $Imgf = D$. By construction, D is an increasing union of groups D_n , where each D_n is generated by one generator. Hence we can define a map

$$0 \mapsto D_n \mapsto M$$

in the following way: let z be a generator of D_n . $z = lz_1$ with z_1 an element of D . $z_1 = lz_2$ with z_2 an element of D . We construct in this way the z_n in a

recursive way. Now, define $z \mapsto (z, z_1, z_2, z_3, \dots)$ and extend to all of D_n . Clearly this map is injective.

Since D is the direct limit of the increasing D_n we obtain that

$$0 \mapsto D \mapsto M$$

If $j : D \mapsto M$ in an injective way, we obtain $jf : Q/Z_{(l)} \mapsto M$ which implies that $jf = 0$. Then $D = \text{Im}gf$ must be included in $\text{Ker}j = 0$, ie $\text{Im}gf = 0$

Therefore $\text{Hom}(Q/Z_{(l)}, \pi_{-1}(L_{K(1)}(K(X_\infty)))) = 0$ as wanted.

From theorem 2.3 we can deduce the same theorem for the localization functor $L_{E(1)}$ and this fact proves the Tate-Thomason Conjecture:

Theorem 2.4: $\pi_{-1}(L_{E(1)}K(X_\infty))$ is reduced

Proof: From [DM94, page 21 Lemma 5.10] we know that there is an homotopy equivalence $L_{E(1)}(K(X_\infty)/l^\nu) \cong L_{K(1)}(K(X_\infty)) \wedge M(Z/l^\nu)$

On the other hand if we consider the exact sequence

$$0 \mapsto Z/l^\nu \otimes \pi_i(L_{K(1)}(K(X_\infty))) \mapsto \pi_i(L_{K(1)}(K(X_\infty))) \wedge M(Z/l^\nu) \mapsto \text{Tor}(Z/l, \pi_{i-1}(L_{K(1)}(K(X_\infty))))$$

By Corollary 2.4, Theorem 2.3 is also verified for $i - 1 = -2$. Using this fact by taking the value $i = -1$ in the exact sequence and using the above homotopy equivalence we obtain that $\pi_{-1}(L_{E(1)}(K(X_\infty)/l^\nu))$ is reduced since both left and right members of the exact sequence are reduced. We then can use theorem 2.3 once again this time with the hypothesis of the reducibility of $\pi_{-1}(L_{E(1)}(K(X_\infty)/l^\nu))$ to conclude that $\pi_{-1}(L_{E(1)}(K(X_\infty)))$ is reduced, as wanted.

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